Technical Notes

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A Low Mach Number Euler Formulation and Application to Time-Iterative LBI Schemes

W. R. Briley,* H. McDonald,† and S. J. Shamroth*
Scientific Research Associates, Inc.
Glastonbury, Connecticut

Introduction

THE usual formulation of the compressible Euler equations encounters singular behavior as the Mach number approaches zero. The Euler equations for a perfect gas are written here in variables that are well behaved and in a form that for constant stagnation enthalpy reduces to the incompressible (constant density) Euler equations as the Mach number approaches zero. The present formulation clarifies how the reduction to incompressible flow occurs and then serves as a guide for a modification of time-iterative linearized block implicit (LBI) schemes (e.g., see Ref. 1) to remove an ill-conditioned behavior at low Mach numbers and thus improve the observed convergence rate.

Incompressible Euler Equations

The incompressible Euler equations can be expressed in nondimensional vector form as

$$\nabla \cdot U = 0 \tag{1}$$

$$\frac{\partial U}{\partial t} + (U\cdot\nabla)U + \nabla\left(\frac{C_p}{2}\right) = 0 \tag{2}$$

where U is the velocity vector, t the time, and C_p a pressure coefficient defined by $C_p \equiv 2(p-p_1)$, where p is pressure and p_1 is an arbitrary (constant) pressure basis. Here and subsequently, all variables are nondimensional, having been normalized by reference quantities denoted by a subscript r. The reference pressure p_r and time t_r have been taken as $\rho_r U_r^2$ and L_r/U_r, respectively, where ρ denotes density and L_r a reference length.

Compressible Euler Equations

The compressible Euler equations can be written in the following form

$$\frac{\partial(\ell n\rho)}{\partial t} + \nabla\cdot U + U\cdot\nabla(\ell n\rho) = 0 \tag{3}$$

$$\frac{\partial U}{\partial t} + \frac{U\partial(\ell n\rho)}{\partial t} + (U\cdot\nabla)U + \rho^{-1}\nabla p = 0 \tag{4}$$

$$\frac{\partial h_0}{\partial t} + U\cdot\nabla h_0 - (\gamma-1)M_r^2\rho^{-1}\frac{\partial p}{\partial t} = 0 \tag{5}$$

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*Vice-President. Member AIAA.

†President. Member AIAA.

The equation of state and definition of stagnation enthalpy h_0 can be expressed for a perfect gas as

$$p = \rho T/\gamma M_r^2 \tag{6}$$

$$h_0 = T + (\gamma-1)M_r^2 q^2/2 \tag{7}$$

where γ is the specific heat ratio, $q^2 \equiv U\cdot U$, $M_r = U_r/c_r$ is the reference Mach number, and the reference stagnation enthalpy has been taken as $c_p T_r$. Here, c_r is the reference sound speed defined by $c_r^2 = \gamma R T_r$, R the gas constant, and c_p the specific heat at constant pressure.

Low Mach Number Formulation

Nonsingular Unsteady Formulation

The state equation (6) indicates that p becomes infinite as $M_r \to 0$ (unless $T \to 0$). To obtain an unsteady formulation which is nonsingular as $M_r \to 0$, a pressure coefficient $\hat{C}_p$ is introduced and defined by

$$\hat{C}_p \equiv 2(p - p_2/\gamma M_r^2) \tag{8}$$

where p_2 is an arbitrary (constant) pressure basis.

Using Eq. (8) to eliminate p in the momentum (4) and energy (5) equations leads to

$$\frac{\partial U}{\partial t} + \frac{U\partial(\ell n\rho)}{\partial t} + (U\cdot\nabla)U + \rho^{-1}\nabla\left(\frac{\hat{C}_p}{2}\right) = 0 \tag{9}$$

$$\frac{\partial h_0}{\partial t} + U\cdot\nabla h_0 - (\gamma-1)M_r^2\rho^{-1}\frac{\partial(\hat{C}_p/2)}{\partial t} = 0 \tag{10}$$

Combining Eqs. (6-8) to eliminate p and T gives

$$\rho h_0 - p_2 = \gamma M_r^2\left[\hat{C}_p + \left(\frac{\gamma-1}{\gamma}\right)\rho q^2\right]\Big/2 \tag{11}$$

If ρ, $\hat{C}_p$, U, and h_0 are taken as dependent variables, then the system of governing equations consists of Eqs. (3) and (9-11), all of which are now well behaved as $M_r \to 0$.

Reduction to Constant-Density Incompressible Formulation

To relate the compressible and incompressible formulations, the limit $M_r \to 0$ is taken by letting $R \to \infty$, which allows U_r, ρ_r, T_r, $\gamma = O(1)$. It can be seen from Eq. (11) that $\rho \to p_2/h_0$ as $M_r \to 0$. Thus, to relate to a constant-density flow as $M_r \to 0$, it is necessary that the stagnation enthalpy h_0 be constant. With h_0 constant as $M_r \to 0$, the terms containing $(\ell n\rho)$ in Eqs. (3) and (9) vanish as $M_r \to 0$, and these equations thus reduce to the incompressible form of Eqs. (1) and (2). If appropriate choices are made for p_1, p_2, and h_0, it follows that $\hat{C}_p \to C_p$ and $\rho \to 1$ as $M_r \to 0$. The compressible and incompressible formulations thus become equivalent.

To guide these choices, it is noted that selecting $p_2 = 1$, $h_0 = 1 + (\gamma-1)M_r^2/2$ implies that $\hat{C}_p = 0$ and $\rho = T = 1$ when $q = 1$ (freestream basis condition for p_1). The choice $p_2 = h_0 = 1$ implies that $\hat{C}_p = 0$ and $\rho = T = 1$ when $q = 0$ (stagnation point basis for p_1). With either choice for p_2 and h_0, $\hat{C}_p \to C_p$ follows as $M_r \to 0$ because $\hat{C}_p$ and C_p satisfy the same governing equations and boundary conditions and are equal at the basis point.

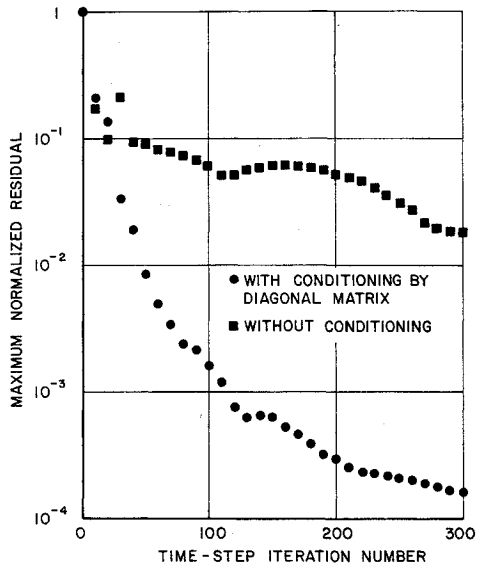

Fig. 1 Convergence rate for Navier-Stokes calculation with $M_r = 0.05$.

Constant Stagnation Enthalpy Formulation for Time-Iterative Algorithms

The condition of constant stagnation enthalpy, which is necessary to obtain the incompressible formulation as $M_r \to 0$, is also a good approximation for a wide range of steady compressible flow problems. The assumption of constant h_0 is valid for inviscid adiabatic flow and is a good approximation for subsonic or transonic viscous flow with a Prandtl number of unity and no heat addition. When h_0 is known, Eq. (11) relates ρ, $\hat{C}_p$, and U and serves to decouple the continuity and momentum equations (3) and (9) from the energy equation (10), which can be omitted from consideration. The system of governing equations then consists of Eqs. (3), (9), and (11), with dependent variables ρ, $\hat{C}_p$, and U. This formulation is very economical for use as a time-iterative approach for computing steady solutions having constant h_0.

By choosing a small value for M_r (say <0.1), the same formulation also provides a useful method for computing a very good approximation of an incompressible flow. Since the definition of h_0 in Eq. (7) applies only to a perfect gas, the temperature is not related to h_0 by Eq. (7) for a constant-density incompressible flow. Consequently, if the incompressible flow is not adiabatic, it is appropriate to solve a separate energy equation for the temperature distribution, once the velocity field has been computed. The Boussinesq assumption for natural convection can be introduced within this same formulation by adding a body force term (which depends only on temperature) to the momentum equation.

Conditioning of Time-Iterative LBI Schemes

A split LBI scheme which is applicable to the present formulation has been described by Briley and McDonald.[1] This algorithm combines a formal time linearization with a Douglas-Gunn alternating-direction implicit (ADI) scheme, and the two-dimensional form of this algorithm can be written in the notation of Ref. 1 as

$$(A + \Delta t \mathfrak{L}_x)(\phi^* - \phi^n) = \Delta t(\mathfrak{D}_x + \mathfrak{D}_y)\phi^n \tag{12a}$$

$$(A + \Delta t \mathfrak{L}_y)(\phi^{n+1} - \phi^n) = A(\phi^* - \phi^n) \tag{12b}$$

Here, n is the time index, ϕ the vector of dependent variables, $\mathfrak{D}_x$ and $\mathfrak{D}_y$ the spatial difference operators associated with spatial (x and y) directions, A the square matrix of coefficients of the time derivatives, and $\mathfrak{L}_x$ and $\mathfrak{L}_y$ are related to $\mathfrak{D}_x$ and $\mathfrak{D}_y$ by the linearization.

To examine the behavior of this algorithm for small M_r, it is helpful to identify $(\hat{C}_p/\rho)$, u, and v as the dependent variables (u and v are the velocity components). The (linearized) algebraic equation (11) is used as needed to relate the dependent variables, but is not treated as a separate governing equation in the algorithm. The A matrix is then given by

$$A = \begin{bmatrix} a & bu & bv \\ au & 1+bu^2 & buv \\ av & buv & 1+bv^2 \end{bmatrix} \tag{13}$$

where $a = \gamma M_r^2 \rho / 2p_2$ and $b = 2a(\gamma - 1)/\gamma$.

Combining Eqs. (12) to eliminate the intermediate result ϕ^* gives

$$(A + \Delta t \mathfrak{L}_x) A^{-1} (A + \Delta t \mathfrak{L}_y)(\phi^{n+1} - \phi^n) = \Delta t(\mathfrak{D}_x + \mathfrak{D}_y)\phi^n \tag{14}$$

For the present formulation, all quantities appearing in Eq. (14) are well behaved as $M_r \to 0$ except A^{-1}, which is singular. This ill-conditioned behavior can be seen from the eigenvalues λ of A, which are given by

$$\lambda = 1,\ c \pm \sqrt{c^2 - a} \tag{15}$$

where $c = (1 + a + bq^2)/2$.

Since a, $b \to 0$ and $c \to 0.5$ as $M_r \to 0$, the eigenvalues of A approach 1, 1, 0, and the condition number becomes infinite.

Since only steady solutions are of interest here, the ill-conditioned matrix A can be replaced by a new matrix $\tilde{A}$ which is well conditioned as $M_r \to 0$. A convenient choice is $\tilde{A} \equiv AD$, where D is a diagonal conditioning matrix whose diagonal entries are a^{-1}, 1, 1.

The eigenvalues λ of $\tilde{A}$ are

$$\lambda = 1,\ d \pm \sqrt{d^2 - 1} \tag{16}$$

where $d = 1 + bq^2/2$.

Since $d \to 1$ as $M_r \to 0$, these eigenvalues of $\tilde{A}$ all approach unity, and the ill-conditioned behavior for small M_r is not present in the modified algorithm (using $\tilde{A}$).

Convergence Rate for Navier-Stokes Calculation with $M_r = 0.05$

The algorithm of Eqs. (12), linearized with respect to ρ, u, and v as the dependent variables, has been tested in a solution of the ensemble-averaged Navier-Stokes equations for turbulent flow in a two-dimensional channel geometry with a 90 degree bend. This test case, which was also considered by Buggeln et al.,[2] has a Reynolds number of 40,000 and a ratio of bend radius to channel width of 2.3. The turbulence model requires solution of the turbulence kinetic energy equation, and a highly stretched (26 × 28) mesh was employed to provide viscous sublayer resolution. Other details are given in Ref. 2.

Two calculations were performed for $M_r = 0.05$, using a time-step selection which provides for the cyclic use of a sequence of steps as described in Ref. 2. These two calculations differ only in whether or not a diagonal conditioning matrix D is included, as described earlier, to remove the ill-conditioned behavior at small M_r. The diagonal matrix D used here has diagonal elements $1/\gamma M_r^2$, 1, 1. The convergence rate is shown in Fig. 1, where the maximum residual in the field is normalized by its value at the start of the calculation. A significant improvement in convergence rate is observed when the algorithm is modified by including the conditioning matrix D.

Acknowledgment

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References

[1]Briley, W. R. and McDonald, H., "Solution of the Multidimensional Compressible Navier-Stokes Equations by a Generalized Implicit Method," *Journal of Computational Physics*, Vol. 24, Aug. 1977, pp. 372-397.

[2]Buggeln, R. C., Briley, W. R., and McDonald, H., "Computation of Laminar and Turbulent Flow in Curved Ducts, Channels and Pipes Using the Navier-Stokes Equations," Scientific Research Associates, Inc., Glastonbury, Conn., Rept. R80-920006-F, Dec. 1980.

Snowing Criteria for Cold Traps

G. M. Harpole*
TRW Space and Technology Group
Redondo Beach, California

Introduction

A COLD trap is a flow channel with cold walls. When a mixture of condensable and noncondensable gases flows between the cold walls, the condensable species diffuse to the walls where they are trapped by condensation and perhaps frozen. In oxygen-iodine lasers, cold traps are used to remove water vapor from an excited oxygen flow. The oxygen is produced in a reaction including an aqueous solution, which leads to water vapor in the flow. This water vapor must be largely removed, because it causes undesirable reactions in the laser cavity. Typically, a maximum water vapor partial pressure requirement is specified; the cold trap surfaces must be cold enough so that the ice on these surfaces has a vapor pressure significantly below the partial pressure requirement. The channel dimensions are determined by considering mass transfer efficiency, gas residence time compared to excited oxygen deactivation times, and pressure drop.

Cold traps also act as heat exchangers and cool the gas as condensable species are removed. The gas cooling tends toward supersaturating the gas, while condensable species removal has the opposite effect. This tradeoff of the local concentration or partial pressure of, say, water vapor p_{H_2O} and the local temperature T can be seen in the supersaturation parameter

$$S = p_{H_2O}/p_{sat}(T)$$

where $p_{sat}(T)$ is the saturation pressure of water, which is correlated (in Pa and K) here as

$$p_{sat}(T) = 133.3\exp(24.1 - 6165/T)$$

When the gas mixture exceeds saturation conditions, fog or ice particles can form in the gas away from the walls. "Heterogeneous nucleation," where water condenses on existing aerosols, occurs with even slight supersaturation, $S>1$. "Homogeneous nucleation" or "spontaneous nucleation" of new fog or snow particles requires greater supersaturation, $S>S_c$, where S_c is the critical supersaturation. In either case, snow is untrapped water that can revaporize in a hot laser cavity.

Temperature and Concentration Profile Analogy

Temperature and concentration gradients are larger in the cross-stream directions than they are in the flow direction. A narrow region in the entrance boundary layer near the channel wall could supersaturate sufficiently to snow, while the gas remains unsaturated at the channel centerline.

An analogy between temperature profiles and concentration profiles will allow a determination of peak supersaturation without actually computing these profiles themselves. The local dimensionless gas temperature is defined as

$$\theta = (T - T_w)/(T_0 - T_w)$$

where T_w is the channel wall temperature and T_0 is the gas inlet temperature. The total pressure in a cold trap typically changes very little, so that the normalized water mole fraction is the same as the dimensionless water partial pressure

$$\phi = \frac{x_{H_2O} - x_{H_2O,w}}{x_{H_2O,0} - x_{H_2O,w}} = \frac{p_{H_2O} - p_{H_2O,w}}{p_{H_2O,0} - p_{H_2O,w}}$$

Equilibrium at the wall demands $p_{H_2O,w} = p_{sat}(T_w)$. Both θ and ϕ vary continuously from 0 to 1 regardless of the channel geometry and regardless of whether the flow is laminar or turbulent. Both θ and ϕ are zero at the wall and far downstream and are unity in a short region near the channel entrance at the channel centerline—they have the same boundary conditions.

The energy conservation equation describing θ and the molar species conservation equation describing ϕ are exactly analogous with the Prandtl number Pr and the Schmidt number Sc, being analogous parameters (see any basic heat and mass transfer text[1]). Define a parameter C by $\phi = C\theta$. If $Pr \simeq Sc$, as for many gases, then $C \simeq 1$. Also, for turbulent flow, eddy diffusivities are the same for both the energy and the molar species equations, and these eddy diffusivities dominate over molecular diffusivities, so that again $C = 1$. Laminar thermal boundary-layer thicknesses are approximately proportional to $Pr^{-0.4}$ (see Ref. 2), so that for laminar boundary layers near the trap inlet $C \simeq (Sc/Pr)^{0.4}$, which for O_2 and H_2O is $C = 0.95$.

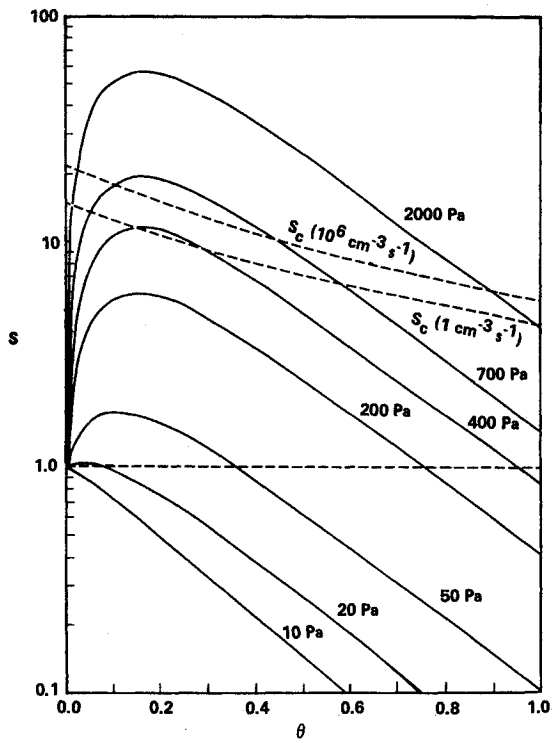

Fig. 1 Supersaturation profiles as a function of θ for various values of p_0^*, the inlet water partial pressure ($T_w = 218$ K, $T_0 = 270$ K).

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*Member, Technical Staff. Member AIAA.